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SPARSE EXPONENTIAL SYSTEMS: COMPLETENESS WITH ESTIMATES

BY

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ABSTRACT

According to A. Beurling and H. Landau, if an exponential system $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ is a frame in L^2 on a set S of positive measure, then Λ must satisfy a strong density condition. We replace the frame concept by a weaker condition and prove that if S is a finite union of segments then the result holds. However, for "generic" S, very sparse sequences Λ are admitted.

1. Introduction

1.1. Given a real sequence of frequencies $\Lambda = \{\cdots < \lambda_0 < \lambda_1 < \cdots\}$, consider the system of exponentials

$$E(\Lambda) := \{e^{i\lambda t}\}_{\lambda\in\Lambda}.$$

Suppose it is complete in the space $L^2(S)$, where $S \subset \mathbb{R}$ is a bounded set of positive measure. This means that any entire function F of exponential type, $F \in L^2(\mathbb{R})$, with Fourier transform supported on S, can be recovered in a unique way from its values on Λ.

The system $E(\Lambda)$ is called **a frame** in $L^2(S)$ if this sampling provides the two-sides estimate

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$$c\|\{F(\Lambda)\}\|_{l^2} \le \|F\|_{L^2(\mathbb{R})} \le C\|\{F(\Lambda)\}\|_{l^2} \tag{1}$$

(with positive constants not depending on F).

The left inequality follows from the "separation condition"

$$\inf \{\lambda_{n+1} - \lambda_n\} > 0, \tag{2}$$

which we always assume.

The right one can be reformulated equivalently as "completeness with l^2 control of coefficients": every f in $L^2(S)$ admits an approximation with arbitrary small error by a linear combination

$$g = \sum_{\lambda \in \Lambda} a(\lambda) e^{i\lambda x}$$

satisfying the condition

$$\|\{a(\lambda)\}\|_2 \le C\|f\|_{L^2(S)}.$$

The case $S = [a, b]$ is classical. In this case, completeness of $E(\Lambda)$ requires some density of Λ. In particular, it cannot satisfy the condition

$$\lambda_{n+1} - \lambda_n \longrightarrow \infty, \quad |n| \to \infty.$$

But it may have "large gaps", meaning that

$$\limsup_{n \to \pm\infty} (\lambda_{n+1} - \lambda_n) = \infty$$

(see, for example, [7]).

However, if $E(\Lambda)$ is a frame in $L^2(S)$ then such gaps are not possible: any sufficiently large interval must contain a number of points from Λ which is proportional to its size. To be more precise, define the "lower Beurling density"

$$D^-(\Lambda) := \lim_{r \to \infty} \frac{n(r)}{r},$$

where $n(r)$ is the minimal number of λ's belonging to a segment of the length r. H. Landau proved [5] that if $E(\Lambda)$ is a frame in $L^2(S)$ then

$$D^-(\Lambda) \ge m(S)/2\pi \tag{3}$$

(by $m(S)$ we denote the Lebesgue measure of S). This deep theorem holds not only for a segment, but for any set S of positive measure.

1.2. In this paper we consider a weaker form of the frame-type condition.

Given $q \geq 2$ we say that the system $E(\Lambda)$ is a (C_q) system in $L^2(S)$ *(complete with l_q estimate of the coefficients)*, if any $f \in L^2(S)$ can be approximated with arbitrary small error by a linear combination

$$g = \sum_{\lambda \in \Lambda} a(\lambda) e^{i\lambda x}, \quad \|\{a(\lambda)\}\|_q \leq K\|f\|_{L^2(S)}.$$

What can be said about the density of Λ in this case? We prove that the answer depends substantially on the nature of the set S. Namely, if it is a segment or a *finite union of segments*, then the estimate (3) still holds.

On the other hand, it turns out that in general *no density condition is necessary.* We construct a sparse Λ such that $E(\Lambda)$ is a (C_q) system in $L^2(S)$ for a generic set S.

"Generic" is understood in the Baire category sense (with respect to the usual distance in the space $\{S\}$). "Sparse" means that the gaps $\lambda_{n+1} - \lambda_n$ tend to infinity quite fast. Any sub-exponential growth is possible.

2. Finite union of segments

2.1. In what follows q is a fixed number, $2 \leq q \leq \infty$.

It is convenient to introduce the (C_q) property in a general setting.

Definition 1: Let H be a Hilbert space. We say that a sequence $\{e_k\} \subset H$ is a (C_q) system in H if, for any vector $f \in H$ and $\epsilon > 0$, there exists a linear combination $g = \sum a_k e_k$ such that

$$\|f - g\| < \epsilon \quad \text{and} \quad \|\{a_k\}\|_q := \left(\sum |a_k|^q\right)^{1/q} \leq K\|f\|,$$

where the constant K does not depend on f and ϵ.

A standard duality argument provides an equivalent form:

LEMMA 1: *$\{e_k\}$ is a (C_q) system in H if and only if there exists $K > 0$ such that the following inequality holds for any $f \in H$:*

$$\|f\| \leq K\|\{\langle f, e_k\rangle\}\|_p \quad (1/p + 1/q = 1). \tag{4}$$

Proof: Indeed, let $\{e_k\}$ be a (C_q) system in H. Take f, $\|f\| = 1$ and $\epsilon > 0$, then choose a corresponding g. We have

$$1 \leq |\langle f, f-g\rangle| + |\langle f, g\rangle| \leq \epsilon + \sum |a_k||\langle f, e_k\rangle| \leq \epsilon + K\|\langle f, e_k\rangle\|_p$$

and (4) follows.

Now, suppose that $\{e_k\}$ is not a (C_q) system in H. Fix K and note that the set of finite linear combinations,

$$G := \{g = \sum a_k e_k : \|\{a_k\}\|_q \le K\},$$

is not dense in the unit ball B of H. So for sufficiently small $\epsilon > 0$ the open convex set $G' := G + \epsilon B$ does not cover $(1-\epsilon)B$, and by the Hahn–Banach theorem, one can choose f, $\|f\| = 1$ and $h \in H$ so that

$$\langle f, h\rangle = 1, \quad \sup_{g\in G'} |\langle g, h\rangle| < 1.$$

This implies

$$\|\{\langle h, e_k\rangle\}\|_p < \frac{1}{K} = \frac{1}{K}|\langle f, h\rangle| \le \frac{1}{K}\|h\|,$$

which contradicts (4). ∎

2.2. Throughout this section S is a finite union of segments:

$$S = \bigcup_{j=1}^{n} [a_j, b_j].$$

Theorem 1: *If Λ is a separated sequence and $E(\Lambda)$ is a (C_q) system in $L^2(S)$, then the inequality (3) holds.*

One says that an entire function F belongs to the *Bernstein class* B_S if it is bounded on the real axes and its Fourier transform is a distribution supported by S.

Λ is called a **sampling sequence** for B_S if any $F \in B_S$ satisfies the inequality

$$\|F\|_{L^\infty(\mathbb{R})} \le C \sup_{\lambda\in\Lambda} |F(\lambda)|,$$

where C is a constant not depending on F.

We will use the following theorem:

Theorem A (Beurling, Landau): *If Λ is a sampling sequence for B_S then*

$$D^-(\Lambda) > m(S)/2\pi.$$

Beurling proved this theorem for a single segment; see [1], p. 346. He also conjectured that the result holds in a much more general situation, in particular for a finite union of segments. This was proved by Landau [5] as a consequence of his theorem, which we stated in the introduction.

Given a (small) $\epsilon > 0$ we denote

$$S_\epsilon := \bigcup_1^n [a_j + \epsilon, b_j - \epsilon].$$

We will prove the following:

Lemma 2: *If $E(\Lambda)$ is a (C_q) system in $L^2(S)$, then Λ is a sampling sequence for B_{S_ϵ}.*

Due to Theorem A this lemma implies Theorem 1.

Proof of Lemma 2: It is enough to consider the case $q = \infty$.

Fix a number $a > 0$ such that $S \subset [-a, a]$. Then, using the separation condition (2), choose $d < 1/2a$ such that

$$\lambda_{n+1} - \lambda_n > d \quad \forall n. \tag{5}$$

Suppose, contrary to the statement of the lemma, that Λ is not a sampling sequence for B_{S_ϵ}. For any $n \geq 1$ choose a function $Q_n \in B_{S_\epsilon}$ such that

$$\begin{aligned} &\|Q_n\|_\infty = 1, \\ &\sup_\Lambda |Q_n(\lambda)| < 1/4n. \end{aligned} \tag{6}$$

Fix $x_n \in \mathbb{R}$ such that $|Q_n(x_n)| > 9/10$. Using Bernstein's inequality for the derivative of an entire function of exponential type, we get that, for every n,

$$1/2 < |Q_n(x_n) - Q_n(\lambda)| \leq \|Q'\|_\infty |x_n - \lambda| \leq a|x_n - \lambda|, \quad \lambda \in \Lambda.$$

So

$$|x_n - \lambda| > d, \quad \lambda \in \Lambda. \tag{7}$$

Now set

$$F_n(x) = Q_n(x)\Big(\frac{\sin\frac{\epsilon}{2}(x - x_n)}{\frac{\epsilon}{2}(x - x_n)}\Big)^2.$$

Clearly F_n belongs to $L^2(\mathbb{R})$ and its Fourier transform, f_n, is the convolution of the distribution $\widehat{Q}_n$ with a continuous function supported on the segment $[-\epsilon, \epsilon]$. So f_n is supported by S. Using the fact that

$$\|F_n\|_\infty \geq |F_n(x_n)| > 9/10$$

we get

$$\|f_n\|_{L^2(S)} \geq (2a)^{-1/2} \|f_n\|_{L^1(S)} \geq C(a) \|F_n\|_{L^\infty(\mathbb{R})} \geq C(a), \tag{8}$$

where $C(a)$ denotes (different) positive constants depending only on a. On the other hand, (6) implies

$$\sum_{\lambda\in\Lambda}|F_n(\lambda)| = \sum_{\lambda\in\Lambda}|Q_n(\lambda)|\Big|\frac{\sin\frac{\epsilon}{2}(\lambda-x_n)}{\frac{\epsilon}{2}(\lambda-x_n)}\Big|^2 \le \frac{1}{n\epsilon^2}\sum_{\lambda\in\Lambda}\frac{1}{|\lambda-x_n|^2}.$$

Using (5) and (7) we conclude that

$$\sum_{\lambda\in\Lambda}|F_n(\lambda)| \le \frac{2}{n\epsilon^2 d^2}\sum_{k\ge 1}\frac{1}{k^2}.$$

The functions $f_n \in L^2(S)$ satisfy the condition (8), and the l_1 norm of the scalar products $\{\langle f_n, e^{i\lambda t}\rangle\}_{\lambda\in\Lambda}$ is arbitrarily small for large n. According to Lemma 1 it follows that $E(\Lambda)$ is not a (C_∞) system in $L^2(S)$. This ends the proof. ∎

3. The generic case

Here we construct a sparse Λ which provides an l_q $(q > 2)$ estimate of coefficients for generic sets. We use a technique from papers [4] and [6].

Consider the space $V = \{S\}$ of measurable sets in a fixed segment $[a, b]$ with the distance

$$\rho(S, S') := m(S \bigtriangleup S').$$

The isometry $S \mapsto \mathbb{1}_S$ (the indicator of S) realizes V as a closed subset of $L^1(a, b)$. This means that V is a complete metric space, so Baire categories may be used. One says that a property holds for a generic set S if it is fulfilled for every S, except for a family of the first category in V.

THEOREM 2: *Given $0 < \epsilon_n \to 0$ $(n \to \infty)$ one can construct a sequence $\Lambda \subset \mathbb{R}^+$ such that*

(i) $\lambda_{n+1}/\lambda_n > 1 + \epsilon_n$, $n = 1, 2, \ldots$.

(ii) *For a generic set $S \in V$ and for any $q > 2$, the exponential system $E(\Lambda)$ is a (C_q) system in $L^2(S)$.*

Note that, if $\{\epsilon_n\}$ decrease slowly, then Λ is very sparse; the gaps may have any sub-exponential growth (in particular, faster than any power).

Condition (i) is sharp. It cannot be replaced by the Hadamardian lacunarity condition, $\lambda_{n+1}/\lambda_n > c > 1$. Indeed, it is well known that in this case the system $E(\Lambda)$ cannot be complete in $L^2(S)$, whenever S is a set of positive measure, Zygmund (1930) (see [8], chapter 5) for integer λ's and Hartman [2] in the general case.

In the proof of the theorem we may assume that $[a,b] = [-\pi,\pi]$. In this case Λ will be a subset of $\mathbb{Z}^+$. Notice that property (ii) for $\Lambda = \mathbb{Z}^+$ follows from [6].

Lemma 3: *Let $\{U_n\} \subset V$ be a sequence of measurable sets, $m(U_n) \to 0$, and $\{d_n\}$ a given sequence of positive numbers. Then a generic set $S \in V$ satisfies the following inequality for infinitely many n's: $m(U_n \cap S) < d_n$.*

Proof: We may assume $d_n \to 0$. Define

$$D_N = \{S \in V : \forall n > N m(U_n \cap S) \geq d_n\}.$$

It is enough to prove that D_N is nowhere dense in V.

For any $S_0 \in V$ and any $r > 0$ consider the ball $B(S_0, r)$. Choose $n > N$ large enough so that $d_n < r/2$ and $m(U_n) < r/2$.

Define $S_1 = S_0 \setminus U_n$. If $S \in B(S_1, d_n)$, then

$$\rho(S, S_0) \leq \rho(S, S_1) + \rho(S_1, S_0) < d_n + m(U_n) < r.$$

So $B(S_1, d_n) \subset B(S_0, r)$. On the other hand,

$$m(S \cap U_n) \leq m(S \setminus S_1) \leq \rho(S, S_1) < d_n,$$

so $B(S_1, d_n) \cap D_N = \emptyset$ and the lemma follows. ∎

We need the following lemma:

Lemma 4: *Given $\delta > 0$ and $q > 2$, one can find a trigonometric polynomial*

$$Q(t) = \sum_{n>0} c_n e^{int},$$

such that

$$mes\{t \in [-\pi,\pi] : |Q(t) - 1| > \delta\} < \delta,$$
$$\|\hat{Q}\|_q := \|\{c_n\}\|_q < \delta.$$

Such a lemma was first proved by Y. Katznelson (1964); see [3], chapter 4, section 2.5. The function constructed there is not an analytic trigonometric polynomial, but it is not difficult to modify the construction in order to get it. Lemma 4 is also a direct consequence of lemma 4.1 and remark 2 on page 382 in [4].

Definition 2: For $l < a \in \mathbb{Z}^+$ denote

$$A(l,a) = (a, 2a, 3a, \ldots, la),$$
$$B(l,a) = \bigcup_{|k| \le l} B_k(l,a), \quad B_k(l,a) = k + A(l, (2l)^{k+l}a).$$

We will use the following properties of the "blocks" $B(l,a)$, which can be checked directly from the definition:

a. $B(l,a) \subset [a-l, \infty)$.

b. The "sub-blocks" $B_k(l,a)$ follow each other, in the sense that, whenever $k_1 < k_2$, the block $B_{k_2}(l,a)$ is situated to the right of the block $B_{k_1}(l,a)$.

c. For any $b_1, b_2 \in B(l,a), b_2 > b_1$, we have

$$b_2 > (1 + d(l))b_1 \quad \text{where } d(l) := \frac{1}{1 + l(2l)^{2l}}.$$

For a trigonometric polynomial $P(t) = \sum c_n e^{int}$ we denote by spec P the set $\{n : c_n \neq 0\}$.

Below, $\mathbb{T}$ is the circle group which is identified with the segment $[-\pi, \pi]$ in the usual way.

Lemma 5: *Let Λ be a sequence of positive integers which contains blocks $B(l_m, a_m)$ with $l_m \to \infty$. Then property* (ii) *above is satisfied.*

Proof: Given $f\colon \mathbb{T} \to \mathbb{C}$ and $M \in \mathbb{Z}^+$, we denote by $f_{[M]}$ the function

$$f_{[M]}(t) = f(Mt), \quad t \in \mathbb{T}.$$

For $S \subset \mathbb{T}$ the set $S_{[M]}$ is defined by the equality

$$\mathbb{1}_{S_{[M]}} = (\mathbb{1}_S)_{[M]}.$$

Denote $\delta_n = 1/n^2$ $(n = 1, 2, \ldots)$. Using Lemma 4, choose a trigonometric polynomial Q_n and a set S_n such that

$$\text{spec}\, Q_n \subset \mathbb{Z}^+, \tag{9}$$

$$\|\widehat{Q}_n\|_{q_n} < \delta_n, \quad q_n = 2 + 1/n, \tag{10}$$

$$|Q_n(t) - 1| \le \delta_n \quad \text{on } S_n, \tag{11}$$

$$m(\mathbb{T} \setminus S_n) < \delta_n. \tag{12}$$

For each n define a number m_n so that

$$l^{(n)} := l_{m_n} > \max\{n, \deg Q_n\}. \tag{13}$$

Denote

$$a^{(n)} := a_{m_n} \quad \text{and} \quad M(n,k) := (2l^{(n)})^{k+l^{(n)}} a^{(n)}.$$

Define

$$\widetilde{S_n} = \bigcap_{|k|\le n} (S_n)_{[M(n,k)]}, \quad U_n = \mathbb{T} \setminus \widetilde{S_n}. \tag{14}$$

Using (12), we have

$$m(U_n) \le \sum_{|k|\le n} m(\mathbb{T} \setminus S_n) \longrightarrow 0.$$

Now consider an arbitrary set S satisfying the condition

$$m(S \cap U_n) < \frac{\delta_n}{(1 + \|Q_n\|_\infty)^2}, \quad \text{for infinitely many } n. \tag{15}$$

According to Lemma 3, this property holds for a generic set on the segment $[-\pi, \pi]$.

We will show that for any $q > 2$ the system $E(\Lambda)$ is a (C_q) system in $L^2(S)$. Fix $q > 2$. It is enough to prove that, for any $f \in L^2(S)$ with $\|f\|_{L^2(S)} = 1$ and any $\epsilon > 0$, one can find a polynomial $P(t)$ with $\operatorname{spec} P \subset \Lambda$ such that $\|\widehat{P}\|_q < 1$ and $\|f - P\| < \epsilon$ (1 can be replaced by any $K > 0$).

Fix f and ϵ. Choose a trigonometric polynomial F such that

$$\|f - F\|_{L^2(S)} < \epsilon/2 \quad \text{and} \quad \|\widehat{F}\|_2 \le 1. \tag{16}$$

Denote deg$F$ by d. Take a large integer n so that the inequality in (15) holds. Define

$$P(t) = \sum_{|k|\le \mathrm{d}} \widehat{F}(k) e^{ikt} (Q_n)_{[M(n,k)]}.$$

Notice, using (9) and (13), that

$$\operatorname{spec}((Q_n)_{[M(n,k)]}) \subset A(l^{(n)}, M(n,k)),$$

so

$$\operatorname{spec}(e^{ikt}(Q_n)_{[M(n,k)]}) \subset B_k(l^{(n)}, a^{(n)}),$$

and from (13) we get

$$\operatorname{spec} P \subset \bigcup_{|k|\le \mathrm{d}} B_k(l^{(n)}, a^{(n)}) \subset B(l^{(n)}, a^{(n)}) \subset \Lambda,$$

provided that $n >$d.

Due to property (b) the spectra of the polynomials $(e^{ikt}(Q_n)_{[M(n,k)]})$ for $k_1 \neq k_2$ are disjoint, so

$$\|\widehat{P}\|_q = \|\widehat{F}\|_q \|\widehat{Q}_n\|_q.$$

Using (10) and (16), we get

$$\|\widehat{P}\|_q \leq \|\widehat{F}\|_2 \|\widehat{Q}_n\|_{q_n} < 1,$$

for any n that satisfies $n > 1/(q-2)$.

Now, using (16) again,

$$\begin{aligned}\|F - P\|^2_{L^2(S)} &= \int_S \Big| \sum_{|k|\leq \mathrm{d}} \widehat{F}(k) e^{ikt}(1 - (Q_n)_{[M(n,k)]}(t)) \Big|^2 \frac{dt}{2\pi} \\ &\leq \sum_{|k|\leq \mathrm{d}} \int_S |1 - (Q_n)_{[M(n,k)]}(t)|^2 \frac{dt}{2\pi}.\end{aligned}$$

Due to (14), each integral does not exceed the sum

$$\int_{(S_n)_{[M(n,k)]}} + \int_{S \bigcap U_n}.$$

We make a change of variables in the first one, and then, estimating it by (11) and the second one by (15), we conclude that

$$\|f - P\|_{L^2(S)} < \epsilon,$$

provided that n is sufficiently large. ∎

Now we can finish the proof of Theorem 2. Given $0 < \epsilon_n \to 0$, we construct by induction a sequence Λ obeying condition (i) of the theorem and satisfying the requirement of Lemma 5.

We can assume that $\{\epsilon_n\}$ is decreasing and that $\epsilon_1 < 1/5$.

Suppose $\{\lambda_n\}_{n\leq N}$ have already been defined. Let l be the maximal integer that satisfies $\epsilon_N < d(l)$, where $d(l)$ was defined in property (c). Keeping in mind property (a), we choose a sufficiently large to ensure $\lambda_{N+1} \geq (1+\epsilon_N)\lambda_N$. We then include all elements of the block $B(l,a)$ as new members of Λ. $\epsilon_n \searrow 0$; so, obviously, the corresponding values of l tend to infinity.

Remark: For a concrete example of a large compact set S, on which a sparse exponential system $E(\Lambda)$ is complete with q-estimate of coefficients (for all $q > 2$), choose $\delta_n = 1/n^3$ and then take S to be the intersection of $\widetilde{S_n}$ $(n > N)$ for sufficiently large N.

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